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VIBRATIONS OF A SEGMENT OF A VARIABLE-LENGTH LONGITUDINALLY-MOVING STRING*

V.V. POPOV

The problem of string vibrations on a section of uniformly changing length was examined in /1/. Transformation belonging to the one-dimensional wave equation group were utilized in /2/ for the solution in the case of an arbitrarily changing length.

An analogous problem is examined below for an arbitrarily moving string. By the successive application of the Galileo and Lorentz transformations it is reduced to a boundary value problem for the wave equation on a section with one moving end, which results in a problem for a segment with fixed ends. The general solution of the problem on waves on a section of variable-length moving string is represented in the form of a sum of natural vibrations. A procedure is proposed for determining the coefficients of the eigenfunction expansion of the solution for initial conditions given in the original variables.

1. The equation describing wave propagation in a string moving at constant velocity v along the x axis has the form

$$u_{tt} + 2vu_{tx} - (c^2 - v^2)\, u_{xx} = 0 \tag{1.1}$$

where c is the wave velocity in the string at rest. We consider the boundary value problem on a segment whose left end is fixed, while the right moves according to the law $x = \Lambda(t)$, $\Lambda(0) = l$, in the general case. In the special case of a uniformly moving end $\Lambda(t) = l + (v + w)t$, where w is the velocity of motion of the end relative to the string material. A taut string rewinding from one coil to another is the simplest model satisfying (1.1) in the interval $[0,\ l + (v + \omega)\, t]$, where the distance between the coils changes. It is assumed that

$$|v| < c,\ |w| < c \tag{1.2}$$

which corresponds to the precritical case.

If the velocities $c_1 = c + v,\ c_2 = c - v$ are introduced, then (1.1) can be written in the form

$$u_{tt} + (c_1 - c_2)\, u_{tx} - c_1c_2u_{xx} = 0$$

This equation describes a one-dimensional system in which the waves in the forward and reverse directions propagate at different velocities /3/. An example might be a hose in which a fluid, considered as an inertial load, flows at a velocity v_0. The equation of the hose vibrations has the form

$$(\rho_0 + \rho_1)u_{tt} + 2v_0\rho_0u_{tx} + (\rho_0v_0^2 - T)\, u_{xx} = 0 \tag{1.3}$$

where ρ_0 and ρ_1 are the linear densities of the fluid and the shell while T is the tension. If the problem is posed for a section, where one of its ends moves at the fluid velocity v_0 (the hose is filled with fluid), then instead of the boundary value problem for (1.3) the problem can be considered for (1.1), where the string parameters are expressed in terms of the hose parameters according to the formulas

$$c^2 = \frac{T}{\rho_0 + \rho_1} - \frac{\rho_0\rho_1v_0^2}{(\rho_0 + \rho_1)^2},\quad v = \frac{\rho_0v_0}{\rho_0 + \rho_1},\quad w = \frac{\rho_1v_0}{\rho_0 + \rho_1}$$

We limit ourselves to problems with the simplest conditions at the segment ends

$$u(t,\ 0) = u(t,\ \Lambda(t)) = 0$$

We take the initial conditions on the segment $[0, l]$ for (1.1) in the form

$$u(0,\ x) = \varphi_0(x),\ u_t(0,\ x) = \psi_0(x) \tag{1.4}$$

2. To solve the problem we construct a transformation that reduces it to a problem for the wave equation on a fixed segment. Carrying out the Galileo transformation $x_1 = x - vt$, $t_1 = t$, we obtain the following equation instead of (1.1) (we omit the subscript 1):

$$u_{tt} - c^2u_{xx} = 0 \tag{2.1}$$

where the interval $[0,\ l + (v + w)\, t]$ transfers into the interval $[-vt,\ l + wt]$, and we have in place of the initial conditions (1.4)

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$$u(0, x) = \varphi(x),\ u_t(0, x) = \psi(x) \tag{2.2}$$
$$\varphi(x) = \varphi_0(x),\ \psi(x) = \psi_0(x) + v\partial\varphi_0(x)/\partial x$$

(The case of an arbitrarily moving end will be examined below).

To obtain segments with one moving end, we apply the Lorentz transformation

$$x' = \frac{x + vt}{\sqrt{1-\beta^2}},\quad t' = \frac{t + \beta x/c}{\sqrt{1-\beta^2}},\quad \beta = \frac{v}{c} \tag{2.3}$$

to (2.1). We then obtain

$$u_{t't'} - c^2 u_{x'x'} = 0,\ x \in [0,\ l' + v't] \tag{2.4}$$

We find the quantities l' and v' by substituting $x = l + wt$ into (2.3) and eliminating the variable t from the relationships obtained

$$v' = \frac{v + w}{1 + \alpha\beta},\quad l' = \frac{l\sqrt{1-\beta^2}}{1+\alpha\beta}\quad \left(\alpha = \frac{w}{c}\right)$$

The first of the formulas, the "relativistic law of addition of velocities", shows that v' remains in the precritical domain since $|v'| < c$ follows from the inequalities (1.2). The second formula shows that the length of the interval experiences an additional change in addition to the "Lorentz contraction" (the radical in the numerator) for $w \neq 0$, because of motion of the end. Being in the domain of applicability of Newtonian mechanics, we regard the passage to t', x' as a formal procedure.

The one-dimensional wave equation allows an infinite group of transformations

$$\xi = f\left(t' + \frac{x'}{c}\right) - f\left(t' - \frac{x'}{c}\right), \tag{2.5}$$
$$\tau = \frac{1}{c}\left[f\left(t' + \frac{x'}{c}\right) + f\left(t' - \frac{x'}{c}\right)\right]$$

where f is an arbitrary function. The change of variables (2.5) transfers the point $x' = 0$ to the point $\xi = 0$. If f is taken in the form (we assume $v' \neq 0$, i.e., $w \neq -v$)

$$f(z) = \frac{L}{\gamma}\ln(v'z + l'),\qquad \gamma = \ln\frac{c + v'}{c - v'}$$

then the interval $[0, l' + v't']$ becomes the interval $[0, L]$, $L = \text{const}$. The general solution of the equation $u_{\tau\tau} - c^2 u_{\xi\xi} = 0$ in $[0, L]$ is represented as the sum of natural vibrations

$$u = \sum_{k=1}^{\infty} \sin\frac{k\pi\xi}{L}\left(a_k\cos\frac{k\pi c\tau}{L} + b_k\sin\frac{k\pi c\tau}{L}\right) \tag{2.6}$$

The variables τ, ξ are expressed in terms of the variable t, x of the original problem on the vibrations of a moving string in a uniformly changing segment with zero boundary conditions by means of the formulas

$$\xi = \frac{L}{\gamma}\ln\frac{\lambda_+}{\lambda_-},\quad \tau = \frac{L}{c\gamma}\ln\left[\frac{(1-\beta)^2}{(1+\alpha\beta)^2}\lambda_+\lambda_-\right] \tag{2.7}$$
$$\lambda_\pm = (v + w)t + l \pm \frac{v + w}{c \mp v}x$$

The asymmetry of the wave phenomena inherent to a moving string is expressed in the appearance of different coefficients for the x coordinate for the forward and reverse waves. For $v = 0$ the solution (2.6) and (2.7) reduces to the solution obtained in /1/.

3. We determine the coefficients a_k and b_k by means of the initial data in the variables t, x in two stages. We first find the functions

$$u|_{t'=0} = \varphi_1(x'),\quad u_{t'}|_{t'=0} = \psi_1(x'),\quad x' \in [0, l'] \tag{3.1}$$

in terms of the functions φ and ψ given in the segment $[0, l]$ for $t = 0$, and then we apply the algorithm to find the coefficients for a problem with one moving end.

In the t, x plane (see the sketch, where the case $v > 0,\ w > 0$ is shown), the space-like line OB with equation $t = -\beta x/c$ corresponds to the equation $t' = 0$. The lines OE and AB correspond to moving boundaries and have the equations $x = -vt$ and $x = l + wt$. The initial conditions should be carried over from the segment OA to the segment OB, i.e., the reverse development of the wave process in time should be traced in the triangle AOB. The point B ($t_B' = 0$, $x_B' = l'$) in the t, x plane has the coordinates $t_B = -\beta l c^{-1}(1 + \alpha\beta)^{-1}$, $x_B = l(1 + \alpha\beta)^{-1}$.

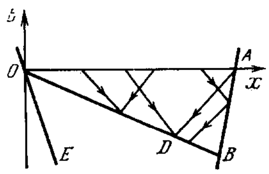
The general solution of Eq.(2.4) has the form

$$u(t', x') = u_1(t' + x'/c) + u_2(t' - x'/c)$$

Since the characteristics $t + x/c = \text{const}$ carry disturbances directly from OA to OB, the formula

$$u_1\left(t+\frac{x}{c}\right)=\frac{1}{2}\varphi(x+ct)+\frac{1}{2c}\int_0^{x+ct}\psi(y)\,dy \tag{3.2}$$

is valid in the whole triangle OAB (the arbitrary constant on the right side of (3.2), which drops out of the final results, is omitted). A formula analogous to (3.2) but with c replaced by - c is valid just for the triangle OAD. The point D is a point of intersection of characteristics of the form $t - x/c = \text{const}$ that pass through the point A and the line OB and has the coordinates $t_D = -\beta l c^{-1}(1+\beta)^{-1}$, $x_D = l(1+\beta)^{-1}$ or $t_D' = 0$, $x_D' = (1-\beta)^{1/2}(1+\beta)^{-1/2}$.



Waves reflected from AB are incident on DB. The reflection condition is determined from the relationship $u(t, l+wt) = 0$ and has the form

$$u_1(t(1+\alpha)+l/c) = -u_2(t(1-\alpha)-l/c)$$

or

$$u_2(z) = -u_1\left(\frac{(c+w)z+2l}{c-w}\right)$$

from which there follows

$$u_2\left(t-\frac{x}{c}\right)=-\frac{1}{2}\varphi(\zeta)-\frac{1}{2c}\int_0^{\zeta}\psi(y)\,dy, \qquad \zeta=\frac{(c+w)(ct-x)+2cl}{c-w}$$

We thus have for the triangle ADB

$$u(t,x)=\frac{1}{2}\varphi(x+ct)-\frac{1}{2}\varphi(\zeta)+\frac{1}{2c}\int_{\zeta}^{x+ct}\psi(y)\,dy \tag{3.3}$$

and for the triangle AOD

$$u(t,x)=\frac{1}{2}\varphi(x+ct)+\frac{1}{2}\varphi(x-ct)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy \tag{3.4}$$

Changing to the variables t', x', setting $t' = 0$, and introducing the notation

$$x_1'=\sqrt{\frac{1-\beta}{1+\beta}}\,x', \qquad x_2'=\sqrt{\frac{1+\beta}{1-\beta}}, \qquad x_3'=\frac{2l}{1-\alpha}-\frac{1+\alpha}{1-\alpha}\sqrt{\frac{1+\beta}{1-\beta}} \tag{3.5}$$

we obtain

$$\varphi_1(x')=\frac{1}{2}\varphi(x_1')+\frac{1}{2}\varphi(x_2')+\frac{1}{2c}\int_{x_2'}^{x_1'}\psi(y)\,dy, \qquad 0\leqslant x'\leqslant x_D' \tag{3.6}$$

For $x_D' \leqslant x' \leqslant l'$ we should replace $\varphi(x_2')$ by $-\varphi(x_3')$ and x_2' by x_3'. Using the formula

$$\frac{\partial}{\partial t'}=\frac{1}{\sqrt{1-\beta^2}}\left(\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right)$$

that follows from (2.3), and expressions (3.3) and (3.4), we have (the prime on a function denotes the derivative with respect to its argument)

$$\psi_1(x')=\frac{1}{\sqrt{1-\beta^2}}\Big[\frac{c}{2}(1-\beta)\varphi'(x_1')-\frac{c}{2}(1+\beta)\varphi'(x_2')+\frac{1}{2}(1-\beta)\psi(x_1')+\frac{1}{2}(1+\beta)\psi(x_2')\Big], \qquad 0\leqslant x'\leqslant x_D' \tag{3.7}$$

For $x_D' \leqslant x' \leqslant l'$ we should replace $\varphi'(x_2')$ by $(1+\alpha)(1-\alpha)^{-1}\varphi'(x_3')$ and $\psi(x_2')$ by $-(1+\alpha)(1-\alpha)^{-1}\psi(x_3')$.

Formulas (3.5)-(3.7) hold even for $w = -v$ when the length of the string segment does not change. The problem of expanding the general natural vibrations solution of (1.1) for a segment of moving string of constant length is of independent interest. In this case the general solution has the same form (2.5) for $\xi = x', \tau = t'$, while the coefficients a_k, b_k are expressed in terms of the functions (3.6) and (3.7) (in which we must put $\alpha = -\beta$, and also, taking (2.2), into account, $\varphi = \varphi_0, \psi = \psi_0 + v\varphi_0'$) by means of the standard formulas

$$a_k=\frac{2}{l'}\int_0^{l'}\varphi_1(y)\sin\frac{k\pi y}{l'}\,dy, \qquad b_k=\frac{2}{l'}\int_0^{l'}\psi_1(y)\sin\frac{k\pi y}{l'}\,dy$$

Returning to the variable-length section, we represent the solution (2.5) in the form

$$u = F\left(\tau + \frac{\xi}{c}\right) - F\left(\tau - \frac{\xi}{c}\right),$$

$$F(z) = \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k \sin\frac{k\pi c}{L} z - b_k \cos\frac{k\pi c}{L} z\right)$$

From (2.5) if follows for an arbitrary function f

$$\tau \pm \frac{\xi}{c} = \frac{2}{c} f\left(t' \pm \frac{x}{c}\right)$$

which yields

$$u = F\left(\frac{2}{c} f\left(t' + \frac{x'}{c}\right)\right) - F\left(\frac{2}{c} f\left(t' - \frac{x'}{c}\right)\right)$$

By satisfying the initial conditions (3.1), we obtain the relationship

$$2F\left(\frac{2}{c} f\left(\pm\frac{x'}{c}\right)\right) = \Phi_{\pm}(x') = \frac{1}{c}\int_0^{x'} \psi_1(y)\,dy \pm \varphi_1(x'), \quad 0 \leqslant x' \leqslant l'$$

(here the arbitrary constant is also omitted). If φ_1 and ψ_1 are continued in the interval $[-l', 0]$ in an uneven manner, we can write

$$\sum_{k=1}^{\infty} a_k \sin\left[\frac{2\pi k}{L} f\left(\frac{x'}{c}\right)\right] - b_k \cos\left[\frac{2\pi k}{L} f\left(\frac{x'}{c}\right)\right] = \Phi_+(x')$$

This relationship holds for $-l' \leqslant x' \leqslant l'$ from which these formulas follow

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \frac{2}{cL}\int_{-l'}^{l'} \Phi_+(y) \begin{Bmatrix} \sin \\ -\cos \end{Bmatrix} \left[\frac{2\pi k}{L} f\left(\frac{y}{c}\right)\right] f'\left(\frac{y}{c}\right) dy \tag{3.8}$$

4. In the general case of a non-uniformly moving end $x' = \Lambda_1(t')$, the construction of a mapping of the segment $[0, \Lambda_1(t')]$ into the segment $[0, L]$ reduces to solving the functional equation

$$f(t' + \Lambda_1(t')/c) - f(t' - \Lambda_1(t')/c) = L \tag{4.1}$$

which is a difficult problem. Following /2/, it is simpler to solve the inverse problem: given the function f, find $\Lambda_1(t')$ from (4.1).

As an illustration, we take

$$f(z) = (az + b)^{-1} \tag{4.2}$$

In the t', x' plane the moving end of the segment describes the hyperbola

$$\frac{a^2 x'^2}{c^2} - \frac{2ax'}{Lc} - (at' + b)^2 = 0$$

Changing to the variables t, x we obtain the law of motion for the end $x = \Lambda(t)$ in implicit form

$$\frac{a^2x^2}{c^2} - \frac{2a^2\beta t x}{c(1-\beta^2)} - \frac{a^2t^2}{1-\beta^2} - \frac{2abt}{\sqrt{1-\beta^2}} - \frac{2ax}{c\sqrt{1-\beta^2}}\left(\frac{1}{L} + \beta b\right) = b^2$$

This will also be a hyperbola. By specifying different functions $f(z)$, a set of exact solutions of the form (2.6) can be obtained for different laws of segment length variation. The coefficients a_k, b_k are here determined by (3.8) by known initial conditions of the form (3.1). If the initial conditions are given in the variables t, x, then they should be carried over to the segment OB (see the sketch) with the reflection condition taken into account on the curvilinear segment AB. This will be the arc of a hyperbola for the case (4.2).

In conclusion, we note that the existence of the expansion (2.6) for the arbitrary vibrations of a moving string in natural vibrations with constant coefficients indicates the presence of stationary characteristics of the non-stationary vibrational processes being considered.

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